A study of the combined effect of thermal radiative transfer and a magnetic field on the gravitational convection of an ionized fluid

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The effect of radiative transfer on the thermal stability of an ionized fluid, in the presence of uniform vertical magnetic field has been examined from hydromagnetic approximations. Two asymptotic cases (i) when the fluid is optically thin and (ii) when it is optically thick, have been examined in detail. The principle of exchange of stabilities and the concept of over-stability have been discussed and the physical conditions under which the former will hold, obtained. Also the conditions as to which type of instability will arise first are derived and it has been shown that in the presence of large magnetic field over-stability arises earlier than convection under the first approximation.

1. Introduction

The problem of the stability of an incompressible fluid enclosed between horizontal surfaces with the lower surface at a higher temperature has been the subject of theoretical investigation by several authors—Rayleigh (1916), Pellew & Southwell (1940). Thompson (1951), Chandrasekhar (1952), Goody (1956), and others. Earlier Bénard (1900, 1901), on the basis of his carefully controlled experiments, had established that a horizontal layer of a fluid when heated from below settles down eventually into a stationary convection pattern in the form of polygonal cells, usually known as cellular convection. Rayleigh explained these results theoretically by showing that instability will set in above a critical value of the dimensionless parameter now known after his name, involving the distance between the surfaces, the temperature gradient, the gravitational acceleration and other physical properties of the fluid like conductivity, viscosity and coefficient of cubical expansion. He also showed that a steady-state solution of the problem does indeed point out the kind of cell formation seen in Bénard's experiments. He had, while doing so assumed the shape of the cells in the horizontal (X, Y)-plane (which is the observation plane in these experiments) to be rectangular. The boundaries of these cells are characterized by either the vertical velocity or its normal derivative being equal to zero, similar to the nodal figures generated in a uniform membrane vibrating freely and held in a horizontal position by being fixed rigidly at its ends. Pellow & Southwell extended the treatment of this problem to a more general cell shape. They also, for the first time, showed the power and usefulness of the application of the variational technique to determine the critical or marginal conditions in this problem. Goody (1956) formally made use of a variational method, extended by Malkus (1954), to take account of the variable temperature gradient in estimating the influence on the stability of a fluid enclosed between two free surfaces of radiative transfer as an extra mode of heat transmission. Spiegel (1960) more recently also undertook to solve the problem together with radiation. The model considered by him, in which the effect of thermal conduction is neglected, is motivated by its possible astrophysical applications. Morton (1957) undertook to demonstrate the importance of viscosity and heat conduction on the departure from equilibrium of a layer of fluid having non-linear temperature gradients. Although these properties had been included in the work of previous authors, Morton explicitly determined the effect of the variation of the Prandtl number σ^* in this problem. He established that the variation of σ^* affects only the initial rate of growth of the disturbance and not the value of the critical Rayleigh number governing marginal stability. He thus claims to have disproved the assertion made by Sutton (1950) in connexion with a different kind of instability observed experimentally by Chandra (1938), that it may be due to a thermal boundary layer near the boundary. Morton also observed that the non-linear temperature gradients do not sensibly effect the state of marginal stability. Due to the importance of this problem in situations of astrophysical and terrestial interest Chandrasekhar extended this analysis to an ionized medium in the presence of a magnetic field, with and without Coriolis forces. His results showed that the magnetic field inhibits convection. About the same time Thompson (1951) studied a similar problem for two rigid surfaces by methods originally employed by Rayleigh and Jeffreys (1928).

There are situations of great physical interest both terrestial and astrophysical in which the role played by magnetic field as well as radiation could be important. Of particular interest are questions relating to stability of high temperature configurations attained by magnetic fields. In an attempt to investigate a stability problem having the effect of thermal radiation as well as magnetic field within its scope, it seems quite expedient to formulate a simple problem to start with. The present paper is such an attempt. It considers the effect of thermal radiative transfer on the stability of an electrically conducting fluid enclosed between two free surfaces in the presence of a vertical magnetic field in the direction of gravitation. We have examined the problem for two approximations of the radiative transfer equation. The various aspects of the problem dealt with are the determination of the critical Rayleigh number for marginal stability as well as over-stability, by a variational method, the physical condition for the applicability of the principle of exchange of stability. The problem has been handled at the hydromagnetic level, that is to say the basic equations for the problem are those of hydrodynamics and Maxwell's equations for the electromagnetic field, in which displacement current is neglected.

2. Basic equations

In order to investigate the radiative transfer effects on the convective stability of a hot electrically conducting fluid enclosed between two horizontal surfaces and heated from below, in the presence of a vertical magnetic field in the same direction as the gravitation, one has to deal with the fundamental equations of magneto hydrodynamics—the continuity, momentum, energy and Maxwell equations—and the integro-differential equation of radiative transfer. These in vector notation are:

$$\frac{\partial \rho}{\partial t}+\nabla.(\rho \mathbf{u}) = 0, \tag{1}$$

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u}\right]-\mu\mathbf{J}\times\mathbf{H} = -\nabla P+\rho\nu\nabla^2\mathbf{u}-\rho\mathbf{g}, \tag{2}$$

$$\frac{\partial T}{\partial t}+(\mathbf{u}.\nabla)T = \kappa\nabla^2 T+\Phi/c_p, \tag{3}$$

$$\left.\begin{aligned} &\operatorname{curl}\mathbf{H} = 4\pi\mathbf{J}, \quad (a) \qquad \operatorname{curl}\mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \quad (b)\\ &\operatorname{div}\mathbf{H} = 0, \quad (c) \qquad \mathbf{J} = \sigma[\mathbf{E}+\mu\mathbf{u}\times\mathbf{H}], \quad (d)\end{aligned}\right\} \tag{4}$$

$$\frac{dI}{ds} = k(B^*-I(s)), \tag{5}$$

$$\Phi = -\int\frac{dI}{ds}d\omega = -4\pi kB^*+k\int I(s)\,d\omega. \tag{6}$$

In these equations, ρ is the density of the fluid, $\mathbf{u}$ the vector velocity, P the pressure, ν the kinematic viscosity, $\mathbf{g}$ the gravitational acceleration, T the temperature, κ the thermal diffusivity, c_p the specific heat per unit volume, Φ the radiative heating per unit volume, I the intensity of radiation at any point, k the absorption coefficient, B^* the Planck function, s and ω are an element of length and solid angle respectively. $\mathbf{J}$ is the current density, $\mathbf{H}$ the intensity of the magnetic field, σ and μ being the electric conductivity and the magnetic permeability respectively, all in electromagnetic units. In the small perturbation method of dealing with the problem of stability one determines the behaviour of the system under investigation when disturbed from its initial state of equilibrium (called the static state) characterized by no convection at all. In the problem in hand we are interested in the motion resulting from heating the lower surface enclosing the fluid. Therefore, in this case, the agency causing perturbation is the temperature. Looked at from this standpoint T in the above differential equations represents the temperature at any point of the fluid in the disturbed state and may be formally put as $T = (T_0+\theta)$, temperature with a subscript zero (i.e. T_0) referring to the static case and θ the perturbation in the temperature field. There being no temperature gradients in the horizontal direction, this may also be put as $T = T^*+\int\beta\,dz+\theta$, z being the vertical co-ordinate and $\beta = -|\beta| = -|dT_0/dz|$ the vertical temperature gradient, and T^* the temperature of the lower plate. Similarly, $P = (p_0+p)$. $\mathbf{u}$ above, then, would be the convective velocity due to the temperature rise θ, $\mathbf{J}$ the electric current resulting from this velocity. This current $\mathbf{J}$ will have an accompanying magnetic field $\mathbf{h}$ say, being related to $\mathbf{H}$ by the relation $\mathbf{H} = (\mathbf{H}_0+\mathbf{h})$, $\mathbf{H}_0$ being the magnetic field intensity in the static state maintained by an external source and in the direction of gravitation. Φ is a function of temperature and hence may be similarly split up as $\Phi = \Phi_0+\phi$, Φ_0 being the radiative heating in the static state and ϕ that due to the perturbation θ. The differential equations for the static state are readily derived. Since $\mathbf{J}$

and $\mathbf{u}$ are zero, the momentum equation leads to the hydrostatic distribution of pressure given by

$$\nabla P_0 = \partial P_0/\partial z = -\rho \mathbf{g}, \tag{7}$$

there being no variation of P_0 in the horizontal (X, Y)-plane. The conduction equation for the static state is

$$\kappa \nabla^2 T_0 + \frac{\Phi_0}{c_p} = \kappa \frac{d^2 T_0}{dz^2} + \frac{\Phi_0}{c_p} = 0, \tag{8}$$

the other two components of ∇^2 being zero. This equation shows that β, the vertical temperature gradient, must be a function of z in the presence of radiative transfer effects, contrasting with the usual case in the absence of Φ_0, where β is constant. The solution of (8) has to be obtained with the help of (6). Goody (1956) has done this with the help of the Milne–Eddington approximation. This model though good enough in the main body of the fluid is not strictly valid near the boundaries. His solution is quoted later in the paper.

Proceeding to solve the differential equations from (1) to (6) some simplifications have to be made, the most important of them being that of linearization; retaining the dependent variables only up to the first power and neglecting their product or square and higher powers. Again the variation in density is neglected except in so far as it effects the buoyancy term—the so-called Boussinesq approximation. The law of density variation with temperature is taken to be the linear relation

$$\rho = \rho_0\{1 - \alpha(\int \beta\, dz + \theta)\}, \tag{9}$$

α being the coefficient of cubical expansion. Equation (7) reduces to

$$dp_0/dz = -\rho_0 g\{1 - \alpha \int \beta\, dz\}, \tag{10}$$

θ being zero for the initial static case. Simplifying the momentum equation with the help of Maxwell's relations and equation (10), we have in terms of the vertical components w and h,

$$\frac{\partial w}{\partial t} = \frac{\mu \mathbf{H}_0}{4\pi\rho_0}\frac{\partial h}{\partial z} + \gamma\theta + \nu\nabla^2\omega - \frac{\partial \chi^*}{\partial z}, \tag{11}$$

where

$$\chi^* = \left(\frac{P}{\rho_0} + \frac{\mu\,|\mathbf{H}|^2}{8\pi\rho_0}\right). \tag{12}$$

Again starting with (4*b*) making use of the Ohm's law (4*d*) and the equation (4*c*) together with the relation $\nabla \cdot \mathbf{u} = 0$ derived from equation (1) after neglecting the variation of density, we obtain for h

$$\frac{\partial h}{\partial t} = H_0 \frac{\partial w}{\partial z} + \eta \nabla^2 h. \tag{13}$$

A similar equation corresponding to (3) is

$$\frac{\partial \theta}{\partial t} = -\beta w + \kappa \nabla^2 \theta + \phi/c_p. \tag{14}$$

Operating on (12) by ∇ . and using $\nabla \cdot \mathbf{h} = 0$ and $\nabla \cdot \mathbf{u} = 0$ we obtain

$$\nabla^2 \chi^* = \gamma \frac{\partial \theta}{\partial z}. \tag{15}$$

Equations (11), (13), (14) and (15) are the linearized equations of the problem for the non-steady case.

The boundary conditions will depend upon whether the two surfaces at the boundary are free or rigid. Since we are solving for two free surfaces the boundary conditions would be

$$\left.\begin{aligned} &\theta = w = 0 \quad \text{at} \quad z = 0 \text{ and } z = d, && \text{(i)}\\ &\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \text{ and } z = d, && \text{(ii)}\\ &\frac{\partial h}{\partial z} = \frac{\partial^3 h}{\partial z^3} = 0 \quad \text{at} \quad z = 0 \text{ and } z = d. && \text{(iii)} \end{aligned}\right\} \tag{16}$$

The boundary conditions (16, iii) are obtainable from (16, i) and the differential equation (13) in h.

3. The equations for the case of marginal stability

The conditions under which the principle of exchange of stabilities is applicable are derived later. We obtain in this section the equations governing marginal stability. It is characterized by $\partial/\partial t = 0$. The equations (11) to (14) therefore become

$$\nu \nabla^2 w = \frac{\partial \chi^*}{\partial z} - \gamma\theta - \frac{\mu H_0}{4\pi\rho_0}\frac{\partial h}{\partial z}, \tag{17}$$

$$H_0 \frac{\partial w}{\partial z} = -\eta \nabla^2 h, \tag{18}$$

$$\beta w = \kappa \nabla^2 \theta + \phi/c_p. \tag{19}$$

These equations can be useful only after ϕ has been expressed as a function of temperature. In general ϕ satisfies an integro-differential equation of the form (6). However, this equation reduces simply in two asymptotic cases. These are characterized by $k^{-1} \gg$ or $\ll$ some characteristic length in the problem (optically thin and optically thick cases). In the first case, intuitive arguments lead to $\Phi = 4\pi k\sigma_0 T^4 + \text{const.}$, whereas in the second, a formal series expansion in terms of k^{-1} (or path length defined as the integral of k over length) and neglect of the terms higher than the first gives

$$\Phi = \frac{4\pi}{3k}\sigma_0 \nabla^2 T^4,$$

σ_0 being Stefan's constant (see Chandrasekhar 1957; Goody 1956; Murgai 1962). If in this problem we regard a, the cell size defined later, as a typical representative length, we have for ϕ after operating with ∇_1^2

$$\nabla_1^2 \phi = -4\pi k S^* \nabla_1^2 \theta \quad (k^2 d^2 \ll a^2) \tag{20}$$

(called hereafter approximation (a)),

and
$$\nabla_1^2 \phi = \frac{4\pi}{3k} S^* \nabla^2 \nabla_1^2 \theta \quad (k^2 d^2 \gg a^2) \tag{21}$$

(called hereafter approximation (b)),

where $\nabla_1^2 = \partial^2/\partial x^{*2} + \partial^2/\partial y^{*2}$ and $S^* = 4\pi\sigma_0(T_0+\theta)^3$,

assumed to be constant. Operating on (17) twice by ∇^2 and on (19) by ∇_1^2, making use of (15) and (18) and eliminating θ, we have two equations in w corresponding to the two forms of ϕ given by equations (20) and (21).

$$\nabla^6 w = \frac{4\pi k S^*}{\kappa c_p}\nabla^4 w - \frac{4\pi k S^* \mu^2 H_0^2 \sigma}{\kappa c_p \rho_0 \nu}\frac{\partial^2 w}{\partial z^2} + \frac{\gamma|\beta|}{\nu\kappa}\nabla_1^2 w + \frac{\mu^2 H_0^2 \sigma}{\rho_0 \nu}\frac{\partial^2}{\partial z^2}\nabla^2 w, \tag{22}$$

and
$$\left(1+\frac{4\pi S^*}{3k\kappa c_p}\right)\nabla^6 w = \frac{\mu^2 H_0^2 \sigma}{\rho_0 \nu}\left(1+\frac{4\pi S^*}{3k\kappa c_p}\right)\frac{\partial^2}{\partial z^2}\nabla^2 w + \frac{\gamma|\beta|}{\nu\kappa}\nabla_1^2 w. \tag{23}$$

These partial differential equations are solved by the method of separation of variables by putting $w = f(x^*, y^*)\,W(z)$, where

$$\nabla_1^2 f = -\frac{a^2}{d^2} f. \tag{24}$$

The differential equation (24) is a Helmholtz equation and is similar to that satisfied by the displacement of a uniform membrane vibrating freely in a horizontal plane. The dimensionless number a is characteristic of the cell shape and size. The simple artifice embodied in the equation (24) seems quite an important point in the theory for this problem. Defining a dimensionless variable $\zeta = (z/d) - \frac{1}{2}$ we have

$$\frac{\partial^2}{\partial z^2} = \frac{1}{d^2}\frac{\partial^2}{\partial \zeta^2} = \frac{D^2}{d^2}, \quad \text{say},$$

$$\nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2} = \frac{D^2 - a^2}{d^2}.$$

With this, equations (22) and (23) become, with the help of (24),

$$[(D^2-a^2)^2 - QD^2]\,(D^2 - a^2 - 3k^2 d^2 \chi)\,w = -Ra^2 \beta w/\bar{\beta}, \tag{25}$$

$$(D^2-a^2)\,[(D^2-a^2)^2 - QD^2]\,(1+\chi)\,w = -Ra^2\beta w/\bar{\beta}, \tag{26}$$

where $\chi = 4\pi S^*/3k\kappa c_p$, $Q = \mu^2 H_0^2 \sigma/\rho_0 \nu$ and $R = \gamma\bar{\beta}\,d^4/\nu\kappa$ is the Rayleigh number, $\bar{\beta}$ being the average temperature gradient.

The boundary conditions in terms of ζ become for free surfaces

$$\left.\begin{aligned} w &= 0 && \text{for} \quad \zeta = \pm\tfrac{1}{2}, \\ Dh = D^3 h &= 0 && \text{for} \quad \zeta = \pm\tfrac{1}{2}, \\ D^2 w = D^4 w &= 0 && \text{for} \quad \zeta = \pm\tfrac{1}{2}. \end{aligned}\right\} \tag{27}$$

The last conditions are obtained from the fact that the temperature is constant at the boundaries.

Goody† (1956) has given the following solution for $\beta/\bar{\beta}$ in the initial static case, namely,

$$\beta/\bar{\beta} = L\cosh\lambda\zeta + M, \tag{28}$$

† This is the correct form of $\beta/\bar{\beta}$ and differs slightly in the expressions for L and M from that given in Goody's paper. We have corresponded with Prof. Goody on this point.

L and M being constants given by

$$L = \chi\left[\frac{2\chi}{\lambda}\sinh\tfrac{1}{2}\lambda+\tfrac{1}{2}(3+3\chi)^{\frac{1}{2}}\sinh\tfrac{1}{2}\lambda+\cosh\tfrac{1}{2}\lambda\right]^{-1},$$

$$M = \frac{L}{\chi}[\tfrac{1}{2}(3+3\chi)^{\frac{1}{2}}\sinh\tfrac{1}{2}\lambda+\cosh\tfrac{1}{2}\lambda],$$

where $$\lambda^2 = 3k^2d^2(1+\chi).$$

Strictly speaking one should, and in principle one can, solve equations (25), (26) together with (28) and the boundary conditions (27). This procedure will give for fixed values of the parameters Q, χ and a^2, a series of values of R from which one could find the minimum. The variation of this minimum value with a^2 would give the critical value R_C of the Rayleigh number. This sequence of steps would have to be carried out for an expected range of Q and χ. This would obviously become quite laborious. The labour involved is considerably reduced by the application of a variational procedure to such problems as first established by Pellew & Southwell (1940) and later extended by Malkus (1954). This procedure is based on the fact that the true solution of the differential equation leads to the minimum value of R. Therefore if one chooses a function satisfying the boundary conditions but not necessarily the differential equation and containing a few arbitrary parameters, one can find an approximate value of $R_{\min}$ by minimizing with respect to them the expression for R. The accuracy of this minimum value can be improved by increasing the number of parameters in the trial function. However, a comparison of the value found from an exact solution and that obtained by this procedure, as done for example by Pellew & Southwell, shows that one or two such parameters are good enough for a fairly accurate value of the Rayleigh number. The critical value R_C is obtained here by another minimization procedure. The various steps are explained below. However, for a trial function chosen to satisfy the boundary conditions for two free surfaces, the arbitrary parameter mentioned above does not appear in the final expression for R. m in equation (34) is a parameter which can however take only integral values.

4. Variational procedure and the determination of R_C

(a) Optically thin case

Multiplying equation (25) on both sides by w and integrating from $-\frac{1}{2}$ to $\frac{1}{2}$ we find

$$R = \frac{1}{a^2I_5}[I_1+3k^2d^2\chi I_2+QI_3+3k^2d^2\chi QI_4] = \frac{I}{a^2I_5}, \tag{29}$$

where

$$\left.\begin{aligned}
I_1 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}[(D^3w)^2+3a^2(D^2w)^2+3a^4(Dw)^2+a^6w^2]\,d\zeta,\\
I_2 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}[(D^2w)^2+2a^2(Dw)^2+a^4w^2]\,d\zeta,\\
I_3 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}[(D^2w)^2+a^2(Dw)^2]\,d\zeta,\\
I_4 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}(Dw)^2\,d\zeta,\\
I_5 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}(\beta/\bar{\beta})\,w^2\,d\zeta.
\end{aligned}\right\} \tag{30}$$

Consider now the effect on R of an arbitrary variation in w compatible with the boundary conditions (27). We have

$$\delta R = \frac{1}{a^2 I_5}\left[\delta I_1 + 3k^2 d^2 \chi \delta I_2 + Q\delta I_3 + 3k^2 d^2 \chi Q \delta I_4 - \frac{I}{I_5}\delta I_5\right], \tag{31}$$

δI_i denoting the corresponding variation in I_i. Integrating by parts and making use of the boundary conditions we have after substituting in (31),

$$\delta R = \frac{-2}{a^2 I_5}\int_{-\frac{1}{2}}^{\frac{1}{2}} \delta w[Q^* w + Ra^2(\beta/\bar{\beta})\, w]\, d\zeta, \tag{32}$$

Q^* standing for the operator in the left-hand side of (25) operating on w. Hence for a small but finite δw, $\delta R \equiv 0$ if

$$Q^*\omega + Ra^2(\beta/\bar{\beta})\,\omega = 0,$$

i.e. if the differential equation (25) is satisfied. The converse of this statement is also true.

(b) Optically thick case

The corresponding expression for R obtainable from (26) by proceeding along similar lines is

$$R = \frac{1+\chi}{a^2}\frac{I_1 + QI_3}{I_5}. \tag{33}$$

The same kind of analysis can be carried out to prove that R given by (26) is a minimum.

In estimating the critical value of the Rayleigh number R_C we choose the following trial function satisfying the boundary conditions,

$$w = \text{const.}\sin m\pi(\zeta + \tfrac{1}{2}), \tag{34}$$

m being an integer. Substituting (34) in (29) we get after integration

$$R = \frac{(m^2\pi^2 + a^2 + 3k^2 d^2 \chi)\,[(m^2\pi^2 + a^2)^2 + Qm^2\pi^2]}{\left[M + \dfrac{8m^2\pi^2}{\lambda(\lambda^2 + 4m^2\pi^2)}\, L\sinh\tfrac{1}{2}\lambda\right] a^2}. \tag{35}$$

Let $a^2 = m^2\pi^2 x$. Then (35) becomes

$$R = \frac{m^4\pi^4}{D_1 x}(1 + x + 3k^2 d^2 \chi_1)\,[(1+x^2) + Q_1], \tag{36}$$

where

$$\chi_1 = \chi/\pi^2, \quad Q_1 = Q/\pi^2,$$

$$D_1 = \left[M + \frac{8m^2\pi^2}{\lambda(\lambda^2 + 4m^2\pi^2)} L\sinh\tfrac{1}{2}\lambda\right].$$

For a given a^2 or x, instability will first set in for the lowest mode $m = 1$. Therefore we have

$$R = \frac{\pi^4}{D_1 x}[(1+x) + 3k^2 d^2 \chi_1]\,[(1+x)^2 + Q_1]. \tag{37}$$

For R to be minimum, $dR/dx = 0$, or

$$2x^3 + 3(1 + k^2 d^2 \chi_1)\,x^2 = (1+Q_1)(1 + 3k^2 d^2 \chi_1). \tag{38}$$

The positive root of (38) when substituted in (37) will give the critical value of R.

S no.	k^2d^2	λ	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}
			$Q = 0$		$Q = 600$		$Q = 1500$	
1	$10^{-6}\times 3.33$	10^{-1}	4.9314	$10^{-1}\times 9.9894$	10×2.6729	10×1.5112	10×3.7486	10×3.4027
2	$10^{-4}\times 3.33$	1	5.0800	1.0916	10×2.7587	10×1.5898	10×3.8689	10×3.4036
3	$10^{-2}\times 3.33$	10	8.5380	10×1.9144	10×5.2946	$10^2\times 1.2422$	10×7.6148	$10^2\times 2.2034$
4	3.33	10^2	—	—	10×7.8879	$10^3\times 4.5521$	$10^2\times 1.2099$	$10^3\times 6.8769$
5	$10^2\times 3.33$	10^3	—	$10^3\times 1.0010$	—	—	—	—
6	$10^4\times 3.33$	10^4	—	$10^3\times 1.0010$	—	$10^4\times 1.5136$	—	$10^4\times 3.2786$
7	$10^6\times 3.33$	10^5	—	$10^3\times 1.0010$	—	$10^4\times 1.5136$	—	$10^4\times 3.2786$
8	$10^8\times 3.33$	10^6	—	$10^3\times 1.0010$	—	$10^4\times 1.5136$	—	$10^4\times 3.2786$
9	$10^{10}\times 3.33$	10^7	—	$10^3\times 1.0010$	—	$10^4\times 1.5136$	—	$10^4\times 3.2786$
			$Q = 3000$		$Q = 6000$		$Q = 10000$	
1	$10^{-6}\times 3.33$	10^{-1}	10×4.8204	10×6.1918	10×6.1790	$10^2\times 1.1343$	10×7.4042	$10^2\times 1.8213$
2	$10^{-4}\times 3.33$	1	10×4.9760	10×6.4502	10×6.3782	$10^2\times 1.1780$	10×7.6432	$10^2\times 1.8879$
3	$10^{-2}\times 3.33$	10	10×9.9552	$10^2\times 3.6402$	$10^2\times 1.2942$	$10^2\times 5.9319$	$10^2\times 1.5647$	$10^2\times 8.7778$
4	3.33	10^2	$10^2\times 1.6925$	$10^3\times 9.7034$	$10^2\times 2.3760$	$10^4\times 1.3405$	$10^2\times 3.0471$	$10^4\times 1.7264$
5	$10^2\times 3.33$	10^3	—	—	—	—	—	—
6	$10^4\times 3.33$	10^4	—	$10^4\times 6.0491$	—	$10^5\times 1.1362$	—	$10^5\times 1.8243$
7	$10^6\times 3.33$	10^5	—	$10^4\times 6.0491$	—	$10^5\times 1.1362$	—	$10^5\times 1.8243$
8	$10^8\times 3.33$	10^6	—	$10^4\times 6.0491$	—	$10^5\times 1.1362$	—	$10^5\times 1.8243$
9	$10^{10}\times 3.33$	10^7	—	$10^4\times 6.0491$	—	$10^5\times 1.1362$	—	$10^5\times 1.8243$

TABLE 1. The values of k^2d^2, a^2, R_C/R_{OOC} for different Q and $\chi = 10^3$.

S no.	k^2d^2	λ	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}
			$Q = 0$		$Q = 600$		$Q = 1500$	
1	$10^{-9}\times 3.33$	10^{-1}	4.9314	$10^{-1}\times 9.9890$	10×2.6730	10×1.5092	10×3.7490	10×3.2730
2	$10^{-7}\times 3.33$	1	5.0890	1.0920	10×2.7580	10×1.5903	10×3.8690	10×3.4243
3	$10^{-5}\times 3.33$	10	8.5330	10×2.4943	10×5.2956	$10^2\times 1.6103$	10×7.6160	$10^2\times 2.8803$
4	$10^{-3}\times 3.33$	10^2	9.8446	$10^4\times 1.3796$	10×7.6880	$10^4\times 5.9099$	$10^2\times 1.2045$	$10^4\times 8.9558$
5	$10^{-1}\times 3.33$	10^3	9.8500	$10^5\times 1.9891$	10×7.7540	$10^5\times 8.7313$	$10^2\times 1.2199$	$10^6\times 1.3246$
6	10×3.33	10^4	—	—	—	—	—	—
7	$10^3\times 3.33$	10^5	—	$10^6\times 1.0010$	—	$10^7\times 1.5121$	—	$10^7\times 3.2753$
8	$10^5\times 3.33$	10^6	—	$10^6\times 1.0010$	—	$10^7\times 1.5121$	—	$10^7\times 3.2753$
9	$10^7\times 3.33$	10^7	—	$10^6\times 1.0010$	—	$10^7\times 1.5121$	—	$10^7\times 3.2753$
			$Q = 3000$		$Q = 6000$		$Q = 100000$	
1	$10^{-9}\times 3.33$	10^{-1}	10×4.8200	10×6.0389	10×6.1790	$10^2\times 1.1343$	10×7.4040	$10^2\times 1.8212$
2	$10^{-7}\times 3.33$	1	10×4.9770	10×6.2944	10×6.3790	$10^2\times 1.1687$	10×7.6450	$10^2\times 2.0355$
3	$10^{-5}\times 3.33$	10	10×9.9580	$10^2\times 4.6451$	$10^2\times 1.2944$	$10^2\times 7.7191$	$10^2\times 1.5030$	$10^3\times 1.1421$
4	$10^{-3}\times 3.33$	10^2	$10^2\times 1.6926$	$10^5\times 1.2429$	$10^2\times 2.3762$	$10^5\times 1.7404$	$10^2\times 3.0472$	$10^5\times 2.2414$
5	$10^{-1}\times 3.33$	10^3	$10^2\times 1.7223$	$10^6\times 1.8295$	$10^2\times 2.4336$	$10^6\times 2.5445$	$10^2\times 3.1405$	$10^6\times 3.3444$
6	10×3.33	10^4	—	—	—	—	—	—
7	$10^3\times 3.33$	10^5	—	$10^7\times 6.0431$	—	$10^8\times 1.1351$	—	$10^8\times 1.8251$
8	$10^5\times 3.33$	10^6	—	$10^7\times 6.0431$	—	$10^8\times 1.1351$	—	$10^8\times 1.8251$
9	$10^7\times 3.33$	10^7	—	$10^7\times 6.0431$	—	$10^8\times 1.1351$	—	$10^8\times 1.8251$

TABLE 2. The values of k^2d^2, a^2, R_C/R_{OOC} for different Q and for $\chi = 10^6$.

S no.	k^2d^2	λ	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}	a^2	R_C/R_{OOC}
			$Q = 0$		$Q = 600$		$Q = 1500$	
1	$10^{-12}\times 3.33$	10^{-1}	4.9300	$10^{-1}\times 9.9889$	10×2.6730	10×1.5092	10×3.7490	10×3.2730
2	$10^{-10}\times 3.33$	1	5.0900	1.0921	10×2.7580	10×1.5899	10×3.8690	10×3.4243
3	$10^{-8}\times 3.33$	10	8.5300	10×1.6273	10×5.2960	$10^2\times 1.6273$	10×7.6160	$10^2\times 2.9107$
4	$10^{-6}\times 3.33$	10^2	9.8400	$10^5\times 1.1436$	10×7.6880	$10^5\times 5.1027$	$10^2\times 1.2045$	$10^5\times 7.7324$
5	$10^{-4}\times 3.33$	10^3	9.8500	$10^6\times 4.4265$	10×7.7540	$10^7\times 1.9621$	$10^2\times 1.2199$	$10^7\times 2.9587$
6	$10^{-2}\times 3.33$	10^4	9.8600	$10^7\times 4.9789$	10×7.7540	$10^8\times 2.2088$	$10^2\times 1.2201$	$10^8\times 3.3296$
7	3.33	10^5	—	—	—	—	—	—
8	$10^2\times 3.33$	10^6	—	$10^9\times 1.0010$	—	$10^{10}\times 1.5121$	—	$10^{10}\times 3.2753$
9	$10^4\times 3.33$	10^7	—	$10^9\times 1.0010$	—	$10^{10}\times 1.5121$	—	$10^{10}\times 3.2753$
			$Q = 3000$		$Q = 6000$		$Q = 10000$	
1	$10^{-12}\times 3.33$	10^{-1}	10×4.8210	10×6.0389	10×6.1790	$10^2\times 1.1343$	10×7.4040	$10^2\times 1.8212$
2	$10^{-10}\times 3.33$	1	10×4.9770	10×6.2944	10×6.3790	$10^2\times 1.1785$	10×7.6450	$10^2\times 1.8886$
3	$10^{-8}\times 3.33$	10	10×9.9580	$10^2\times 4.6942$	$10^2\times 1.2944$	$10^2\times 7.8008$	$10^2\times 1.5030$	$10^3\times 1.1543$
4	$10^{-5}\times 3.33$	10^2	$10^2\times 1.6926$	$10^6\times 1.0732$	$10^2\times 2.3762$	$10^6\times 1.4913$	$10^2\times 3.0472$	$10^6\times 1.9352$
5	$10^{-4}\times 3.33$	10^3	$10^2\times 1.7223$	$10^7\times 4.0890$	$10^2\times 2.4336$	$10^7\times 5.6870$	$10^2\times 3.1405$	$10^7\times 7.2756$
6	$10^{-2}\times 3.33$	10^4	$10^2\times 1.7227$	$10^8\times 4.5985$	$10^2\times 2.4342$	$10^8\times 6.3951$	$10^2\times 3.1415$	$10^8\times 8.1810$
7	3.33	10^5	—	—	—	—	—	—
8	$10^2\times 3.33$	10^6	—	$10^{10}\times 6.0431$	—	$10^{11}\times 1.1351$	—	$10^{11}\times 1.8225$
9	$10^4\times 3.33$	10^7	—	$10^{10}\times 6.0431$	—	$10^{11}\times 1.1351$	—	$10^{11}\times 1.8225$

TABLE 3. The values of k^2d^2, a^2, R_C/R_{OOC} for different Q and for $\chi = 10^9$.

For case (b) the equations corresponding to (37) and (38) are respectively

$$R = \frac{1+\chi}{D_1}\pi^4\frac{(1+x)}{x}[(1+x)^2+Q_1], \tag{39}$$

and

$$2x^3+3x^2 = 1+Q_1. \tag{40}$$

The value of x determined as a root of (39) when substituted in (38) gives

$$R_C = \frac{1+\chi}{D_1}R_{MOC},$$

where R_{MOC} is the critical value of the Rayleigh number with magnetic field but no radiation and is the same as obtained by Chandrasekhar (1952). It is of interest

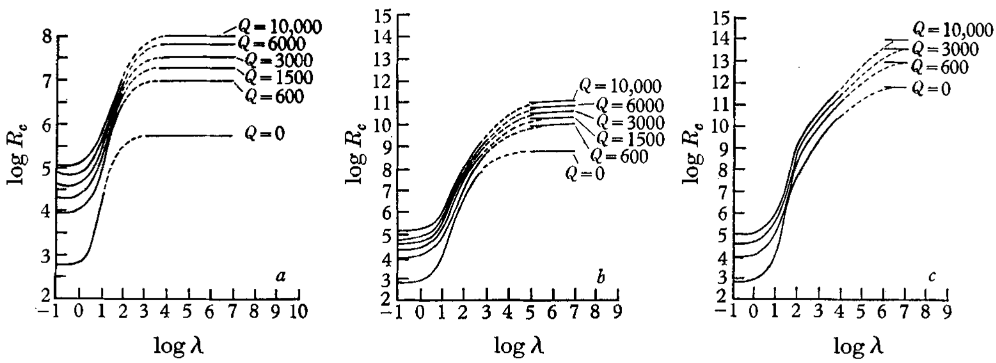

FIGURE 1. Plot of critical Rayleigh number as a function of λ for different values of Hartman number Q given on the curves. The dimensionless quantities λ, χ and Q are characteristic of absorption coefficient and distance between the horizontal planes, temperature in the equilibrium state and magnetic field respectively. The unbroken segments on the left of each curve are given by approximation (a), where $(kd)^2 \ll a^2$, and those on the right by approximation (b), where $(kd)^2 \gg a^2$. The dotted lines joining the full ones represent the interpolation of the two approximations where neither of them holds. (a) $\chi = 10^3$; (b) $\chi = 10^6$; (c) $\chi = 10^9$.

to point out that as χ or $\lambda \to 0$, equations (37) and (39) and also (38) and (40) reduce to two equations obtained by Chandrasekhar and that for χ, λ and Q all $\to 0$ equation (37) or (39) give the expression for the critical Rayleigh number $= \pi^4(1+x)^3/x$ as obtained by Pellew & Southwell (1940). We denote it by R_{OOC}. Tables 1–3 give the values of a^2 and R_C/R_{OOC} for different values of Q and λ, for $\chi = 10^3$, 10^6 and 10^9 respectively. Figure 1 gives a plot of $\log R_C$ versus $\log\lambda$ for the above values of χ and for the values of Q given on the curves, while figure 2 represents $\log R_C$ versus $\log Q$ for the different values of λ given on the curves. The curves for $Q = 0$ are for no magnetic field and are in fact the curves obtained by Goody (1956). The stabilizing effect of an additional magnetic field is obvious. The stabilizing influence of radiative transfer is quite evident from figure 2 where the curves for $\lambda = 10^{-1}$ and 1 coincide with that for $\lambda = 0$, which corresponds to no radiation.

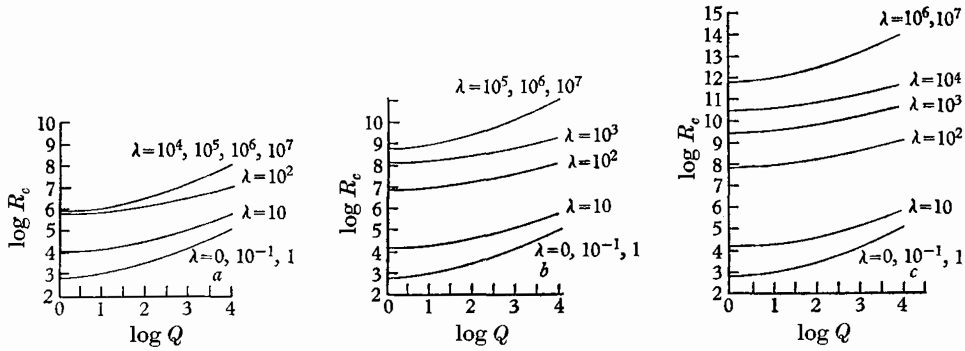

FIGURE 2. Plot of critical Rayleigh number R_c against Hartman number Q for different values of λ given on the curves. The dimensionless quantities λ, χ and Q are characteristic of absorption coefficient and distance between the horizontal planes, temperature in the equilibrium state and magnetic field respectively. The top curve on each diagram is given by approximation (*b*) where $(kd)^2 \gg a^2$, and the others by approximation (*a*) where $(kd)^2 \ll a^2$. (*a*) $\chi = 10^3$; (*b*) $\chi = 10^6$; (*c*) $\chi = 10^9$.

5. The principle of the exchange of stabilities

In this section we propose to examine the physical conditions under which the principle of exchange of stabilities applies in relation to the problem under investigation. The concept of over-stability as described by Chandrasekhar (1961) is also discussed. As pointed out already equations (11) to (14) are the time-dependent equations of the problem. By operating with ∇^2 on equation (11), χ^* can readily be eliminated. The resulting equations become

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\nabla^2 w) &= \frac{\mu H_0}{4\pi\rho_0}\frac{\partial}{\partial z}(\nabla^2 h)+\nu\nabla^4 w+\gamma\nabla_1^2\theta,\\ \frac{\partial\theta}{\partial t} &= -\beta w+k\nabla^2\theta+\phi/c_p\\ \frac{\partial h}{\partial t} &= H_0\frac{\partial w}{\partial z}+\eta\nabla^2 h. \end{aligned}\right\} \tag{41}$$

If the time dependence in these equations is assumed to be like exp (p^*t) the set of equations will lead to

$$\left.\begin{aligned} (p^*-\nu\nabla^2)\,\nabla^2 w &= \frac{\mu H_0}{4\pi\rho_0}\frac{\partial}{\partial z}(\nabla^2 h)+\gamma\nabla_1^2\theta,\\ (p^*-k\nabla^2)\,\theta &= -\beta w+\phi/c_p,\\ (p^*-\eta\nabla^2)\,h &= H_0\frac{\partial w}{\partial z}. \end{aligned}\right\} \tag{42}$$

Substituting in (42) the asymptotic form of ϕ given by equation (20) and eliminating θ we have in terms of the operator D^2 the following equation for case (*a*):

$$[n-\nu(D^2-a^2)]\,[n-\kappa(D^2-a^2)]\,(D^2-a^2)\,w+\frac{4\pi kS^*d^2}{c_p}[n-\nu(D^2-a^2)]\,(D^2-a^2)w$$
$$=\frac{\mu H_0 d}{4\pi\rho_0}\left[n-\kappa(D^2-a^2)+\frac{4\pi kS^*d^2}{c_p}\right](D^2-a^2)\,Dh+\gamma a^2 d^4\beta\omega, \tag{43}$$

where $n = p^* d^2$, and

$$[n - \eta(D^2 - a^2)]\, h = H_0 d D w. \tag{44}$$

With the characteristic-value problem now put in this form one can proceed to investigate the criteria to be satisfied for convection or over-stability to occur. In relation to equation (43) and (44) these respective cases would arise according as (i) n is real and positive or (ii) n is complex, $\mathrm{Re}(n) > 0$ with a finite imaginary part. When $\mathrm{Re}(n) < 0$, whatever be the imaginary part, the disturbance will decay with decreasing amplitude, returning in the limit to the initial static state.

If w_i, w_j are two eigenvalues of w corresponding to n_i and n_j respectively, one can obtain by a series of steps starting from (43) and (44) (see the appendix for details) the equation

$$\begin{aligned}[n_i - n_j]\bigg[(n_i + n_j)\int_{-\frac{1}{2}}^{\frac{1}{2}} &(Dw_i Dw_j + a^2 w_i w_j)\, d\zeta \\ &+ (\nu + \kappa)\int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2 w_i D^2 w_j + 2a^2 Dw_i Dw_j + a^4 w_i w_j)\, d\zeta \\ &+ \frac{\mu}{4\pi\rho_0}(\eta - \kappa)\int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2 h_i D^2 h_j + 2a^2 Dh_i Dh_j + a^4 h_i h_j)\, d\zeta \\ &+ 3k^2 d^2 \chi\kappa \bigg\{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(Dw_i Dw_j - \frac{\mu}{4\pi\rho_0} Dh_i Dh_j\right) d\zeta \\ &+ a^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(w_i w_j - \frac{\mu}{4\pi\rho_0} h_i h_j\right)\bigg\}\, d\zeta\bigg] = 0. \end{aligned} \tag{45}$$

A complex value of n (i.e. p^*) will be associated with a complex value of w and h as will their respective complex conjugates $(\bar{n}, \bar{w}, \bar{h})$ which will satisfy the same equation and boundary conditions. Now we take (n_i, w_i, h_i) as (n, w, h) and (n_j, w_j, h_j) as $(\bar{n}, \bar{w}, \bar{h})$ respectively. Equation (45) then becomes

$$\begin{aligned}\mathrm{Im}(n)\,[2\,\mathrm{Re}(n)\, I_1^* &+ (\nu + \kappa)\, I_2^* + (4\pi\rho_0)^{-1}\mu(\eta - \kappa)\, I_3^* \\ &+ 3k^2 d^2 \chi\kappa\{(I_4^* - (4\pi\rho_0)^{-1}\mu I_5^*) + a^2(I_6^* - (4\pi\rho_0)^{-1}\mu I_7^*)\}] = 0, \end{aligned} \tag{46}$$

where the I^*'s stand for the corresponding integrals and are positive definite. Now if

$$\eta > \kappa, \tag{47}$$

$$I_4^* > \frac{\mu}{4\pi\rho_0} I_5^*, \quad I_6^* > \frac{\mu}{4\pi\rho_0} I_7^*, \tag{48}$$

and $\mathrm{Re}(n)$ is positive, $\mathrm{Im}(n)$ must be zero. This corresponds to the first kind of instability mentioned above, namely convection. The marginal state in this case is the limit of solutions as $n \to 0$ through positive values. That this limiting solution is, in fact, the marginal state for cellular convection can be seen thus. Putting $\mathrm{Im}(n) = 0$ in equation (VI) of the appendix, we get the following quadratic in n:

$$A_1 n^2 + B_1 n + C_1 = 0. \tag{49}$$

A_1 and B_1 are both always positive. As $n \to 0$ through positive values, $C_1 \to 0$ through negative values and one recovers from $C_1 = 0$ the value of the Rayleigh number characterizing the marginal state for convection. For the particular type of boundary conditions this reduces to equation (37). The conditions given by (47) and (48) are thus sufficient conditions for the principle of exchange of stabilities to be valid.

We now consider the case when n is complex. In this case since $\mathrm{Im}(n) \neq 0$, we have

$$2\,\mathrm{Re}(n)\,I_1^* + (\nu+\kappa)\,I_2^* + (4\pi\rho_0)^{-1}\mu(\eta-\kappa)\,I_3^* \\ + 3k^2 d^2\chi\kappa\{(I_4^* - (4\pi\rho_0)^{-1}\mu I_5^*) + a^2(I_6^* - (4\pi\rho_0)^{-1}\mu I_7^*)\} = 0. \quad (50)$$

In order that $\mathrm{Re}(n)$ may be positive, we must have

$$(\nu+\kappa)\,I_2^* + (4\pi\rho_0)^{-1}\mu(\eta-\kappa)\,I_3^* \\ + 3\kappa^2 d^2\chi\kappa\{(I_4^* - (4\pi\rho_0)^{-1}\mu I_5^*) + a^2(I_6^* - (4\pi\rho_0)^{-1}\mu I_7^*)\} < 0. \quad (51)$$

This is, therefore, the necessary criterion for instability to arise through oscillations of increasing amplitude, described as over-stability. That $\mathrm{Im}(n)$ tends to a definite limit as $\mathrm{Re}(n) \to 0$ in this case may be readily proved. In the process of doing this we not only find the minimum value of the Rayleigh number for the onset of overstability but also the frequency of oscillations. Putting the trial function or w as given by (34) into (44) we have

$$h = \frac{H_0\,d\pi}{n+\eta(\pi^2+a^2)}\cos\pi(\zeta+\tfrac{1}{2}). \quad (52)$$

Substituting these values of w and h in (VI) of the appendix, suppressing the subscripts and after some manipulation, we obtain for the case (a):

$$n^3y + n^2\pi^2[(\kappa+\eta+\nu)\,y^2 + 3k^2 d^2\chi_1\kappa y] \\ + n\pi^4[(\eta\kappa+\eta\nu+\nu\kappa)\,y^3 + 3k^2 d^2\chi_1\kappa(\eta+\nu)\,y^2 + Q_1\eta\nu y - \pi^{-6}Ra^2\nu\kappa D_1] \\ + \pi^6\kappa\eta\nu[y^4 + 3k^2 d^2\chi_1 y^3 + Q_1 y^2 + 3k^2 d^2\chi_1 Q_1 y - R\pi^{-6}a^2 y D_1] = 0, \quad (53)$$

where $$y = 1+\frac{a^2}{\pi^2} = 1+x; \quad \chi = \frac{4\pi S^*}{3\kappa k c_p},$$

$\chi_1 = \chi/\pi^2$; $Q_1 = Q/\pi^2$; $n = n_i = n_j$. We write (53) as

$$n^3 + B'n^2 + C'n + D' = 0. \quad (54)$$

Let $n = p' + iq'$. Separating the real and imaginary parts of (54), we have

$$\left.\begin{aligned} p'^3 - (3p'+B')\,q'^2 + B'p'^2 + C'p' + D' &= 0, \quad \text{(i)} \\ \text{and} \qquad q'[3p'^2 - q'^2 + 2B'p' + C'] &= 0, \quad \text{(ii)} \end{aligned}\right\} \quad (55)$$

so that either

$$\left.\begin{aligned} q' &= 0, \quad \text{(i)} \\ \text{or} \qquad 3p'^2 + 2B'p' + C' &= q'^2. \quad \text{(ii)} \end{aligned}\right\} \quad (56)$$

The case $q' = 0$ which corresponds to convection has already been discussed above and the second case corresponds to over-stability. In the marginal state as $p' \to 0$, q' tends to a definite limit, namely

$$\left.\begin{aligned} q'^2 = D'/B' &= C', \quad \text{(i)} \\ \text{or} \qquad B'C' &= D'. \quad \text{(ii)} \end{aligned}\right\} \quad (57)$$

Equation (57) is thus the condition for over-stability just to arise. Substituting the values of B', C' and D' in (57), we obtain after some simple algebra

$$R = \frac{\pi^4}{D_1} \frac{(\eta+\nu)(\eta+\kappa)}{\nu\kappa} \frac{(1+x)^2}{x} \left[(1+x) + 3k^2 d^2 \chi_1 \frac{\kappa}{\eta+\kappa} + \frac{Q_1 \eta\nu}{(\eta+\kappa)(\nu+\kappa)\{(1+x)+3(\nu+\kappa)^{-1} k^2 d^2 \chi_1 \kappa\}} \right]. \tag{58}$$

In order to obtain the critical value of the Rayleigh number, the expression (58) can be minimized in a similar manner as done for the case of marginal stability. The equation giving x in this case is

$$[2x^2 + x(A+1) - (A+1)](1+x+B)^2 = T_1[(B+1) - x(B-1)], \tag{59}$$

where
$$A = 3k^2 d^2 \chi_1 \kappa/(\eta+\kappa),$$
$$B = 3k^2 d^2 \chi_1 \kappa/(\nu+\kappa),$$
$$T_1 = Q_1 \eta\nu/(\eta+\kappa)(\nu+\kappa).$$

The frequency of oscillations q' in the marginal state is given by

$$q'^2 = \frac{D'}{B'} = C',$$

or
$$q' = \frac{\pi^2}{d^2}\left[\frac{\kappa\eta\nu\{y^3 + 3\kappa^2 d^2 \chi_1 y^2 + Q_1 y + 3\kappa^2 d^2 \chi_1 Q_1 - \pi^{-4} R D_1 (y-1)\}}{(\kappa+\eta+\nu)y + 3\kappa^2 d^2 \chi_1 \kappa}\right]^{\frac{1}{2}}$$
$$= \frac{\pi^2}{d^2}[(\eta\kappa+\eta\nu+\nu\kappa)y^2 + 3k^2 d^2 \chi_1 \kappa(\eta+\nu)y - Q_1 \eta\nu - \pi^{-4} R\nu\kappa y^{-1}(y-1) D_1]^{\frac{1}{2}}. \tag{60}$$

The equation corresponding to (43) in case (b) is

$$[n - \nu(D^2-a^2)][n - \kappa_1(D^2-a^2)](D^2-a^2)w = \frac{\mu H_0 d}{4\pi\rho_0}[n - \kappa_1(D^2-a^2)](D^2-a^2)h + \nu\gamma a^2 d^4 \beta w, \tag{61}$$

where $\kappa_1 = \kappa(1+\chi)$. One obtains (see the appendix) from (61) the following equation corresponding to (46) for case (a):

$$\mathrm{Im}(n)[2\,\mathrm{Re}(n) I_1^* + (\nu+\kappa_1) I_2^* + (4\pi\rho_0)^{-1}\mu(\eta-\kappa_1) I_3^*] = 0. \tag{62}$$

Following arguments similar to those in case (a) it may be shown that for the principle of exchange of stabilities to hold the condition

$$\eta > \kappa_1 \tag{63}$$

is a sufficient one. For χ or $\lambda \to 0$, equations (47) and (62) reduce to a common equation and (47), (48) and (63) give place to

$$\eta > \kappa, \tag{64}$$

a condition obtained by Chandrasekhar (1952).

The corresponding value of the Rayleigh number for over-stability is

$$R = \frac{1+\chi}{D_1} \pi^4 \frac{(\eta+\nu)(\eta+\kappa_1)}{\nu\kappa_1} \frac{(1+x)}{x} \left[(1+x)^2 + \frac{Q_1 \eta\nu}{(\eta+\kappa_1)(\nu+\kappa_1)}\right] \tag{65}$$

from which a critical value of R may be easily obtained. The frequency of oscillations in the marginal state is given by

$$q' = \frac{\pi^2}{d^2}\left[\frac{\kappa_1\eta\nu\{y^3+Q_1 y - RD_1(1+\chi)^{-1}\pi^{-4}(y-1)\}}{(\kappa_1+\eta+\nu)\,y}\right]^{\frac{1}{2}}$$

$$= \frac{\pi^2}{d^2}[(\eta\kappa_1+\eta\nu+\nu\kappa_1)\,y^2+Q_1\eta\nu-\pi^{-4}R\nu\kappa\, D_1(y-1)/y]^{\frac{1}{2}}. \tag{66}$$

The values of the Rayleigh number given by (65) and (58) and those of frequency of oscillations given by (60) and (66) lead to Chandrasekhar's results for χ or $\lambda \to 0$.

6. Manner of onset of instability

It is of interest to know which type of instability—convection or overstability—will arise first. One can find this information conveniently for very large values of Q only. This will also provide us with the necessary conditions for the validity of the principle of exchange of stabilities.

Let us call $R_C^{(\mathrm{con.})}$ and $R_C^{(\mathrm{O.S.})}$ the limiting critical Rayleigh numbers evaluated from (37) and (58) for $Q \to \infty$.

Case (*a*). From (38), we have

$$x_{\min} \to [\tfrac{1}{2}(1+3\kappa^2 d^2\chi_1)\,Q_1]^{\frac{1}{3}}, \tag{67}$$

$$R_C^{(\mathrm{con.})} \to \pi^4 Q_1/D_1. \tag{68}$$

Again equation (59) may be written as

$$[2x^2+x(A+1)-(A+1)]\frac{(1+x+B)^2}{[1-x(B-1)/(B+1)]} = T_1(B+1). \tag{69}$$

Now three cases arise, namely (i) $B \ll 1$, (ii) $B = 1$ and (iii) $B \gg 1$. The first two are possible for small values of λ (say $\lambda = 10^{-1}, 1$) when in fact radiative transfer effects are quite small and the fluid motion is largely affected by the magnetic field. Case (iii) is possible for large values of λ within the limitations imposed by case (*a*) (say $\lambda = 10$ to 10^4 depending upon the values of χ used). In this case the fluid motion is also affected by radiative transfer.

(i) $B \ll 1$, in this case $(B-1)/(B+1) \sim -1$, so that from (69) when $Q \to \infty$ (i.e. $T_1 \to \infty$), we obtain

$$\left.\begin{aligned} x_{\min} &\to [\tfrac{1}{2}T_1(B+1)]^{\frac{1}{3}},\\ R_C^{(\mathrm{O.S.})} &\to \frac{\pi^4}{D_1}\frac{(\eta+\nu)\,\eta}{(\nu+\kappa)\,\kappa}Q_1.\end{aligned}\right\} \tag{70}$$

(ii) For $B = 1$ and $Q_1 \to \infty$, we obtain from (69)

$$x_{\min} \to (T_1)^{\frac{1}{4}},$$

and thus

$$\left. R_C^{(\mathrm{O.S.})} \to \frac{\pi^4}{D_1}\frac{(\eta+\nu)\,\eta}{(\nu+\kappa)\,\kappa}Q_1.\right\} \tag{71}$$

Now suppose instability as convection arises first which requires

$$R_C^{(\mathrm{con.})} < R_C^{(\mathrm{O.S.})} \quad \text{or} \quad \frac{(\eta+\nu)\,\eta}{(\nu+\kappa)\,\kappa} > 1;$$

this will always be satisfied if $\eta > \kappa$, a condition already obtained by Chandrasekhar (1952). This result is otherwise obvious too. It can easily be shown that the limiting value of the frequency of oscillations in the marginal state for over-stability is not affected by radiation.

(iii) $B \gg 1$. In this case $(B-1)/(B+1) \sim 1$, and from (69) the only positive root for $Q \to \infty$ is

$$x = 1, \tag{72}$$

and does not depend upon T_1 (or Q_1) as is usually the case. And

$$R_C^{(\text{O.S.})} \to \frac{\pi^4}{D_1} \frac{4}{B} \frac{(\eta+\nu)}{(\nu+\kappa)} \frac{\eta}{\kappa} Q_1 \tag{73}$$

for convection to arise first, we have

$$\frac{(\eta+\nu)}{(\nu+\kappa)} \frac{\eta}{\kappa} \frac{4(\nu+\kappa)}{3k^2 d^2 \chi_1 \kappa} > 1,$$

or

$$\frac{(\eta+\nu)}{\kappa} \frac{\eta}{\kappa} > \frac{3k^2 d^2 \chi_1}{4} \sim \frac{\lambda^2}{4\pi^2},$$

which is satisfied if

$$(\eta/\kappa) > \lambda/2\pi. \tag{74}$$

It appears that in this case overstability will arise earlier than convection and we cannot apply the principle of exchange of stabilities.

Case (*b*). In this case it can be shown that $x_{\min}$ varies as $(Q_1)^{\frac{1}{3}}$ both in the case of convection as well as over-stability. But this clearly violates the restriction $k^2 d^2 \gg a^2$ essential for this approximation. It appears that for a given Q and large χ or for a given χ and large Q, fluid motion can be treated at the level of approximation (*a*) only.

7. Discussion

Radiative transfer and magnetic field have a stabilizing influence. Tables 1–3 show the total inhibiting effect for different values of Q. This results from the fact that radiative transfer tends to damp out any motions which may arise due to heat transfer from hotter to colder parts of the fluid. Again the presence of magnetic field will make more difficult the closing down of the streamlines and the consequent cell formation. As expected the size of the Bénard cells for case (*a*) is larger than that in the presence of radiative transfer or a magnetic field alone.

From the curves of $\log R_C$ versus $\log \lambda$ it is clear that for small values of $\lambda = 10^{-1}$, 1, the inhibiting effect is mainly due to the magnetic field and there are no radiative transfer effects on the convective motion. As the value of λ increases within the restrictions imposed by the approximation (*a*) the radiative transfer becomes important and largely contributes to the stability of the fluid. The 'inhibition' by the magnetic field is quite small in this region. This is evident from the fact that curves for different values of Q are crowded in the neighbourhood of the curve for $Q = 0$.

In the case (*b*) the fluid behaves as if having a diffusivity $\kappa_1 = \kappa(1+\chi)$ and the stabilizing effect of the magnetic field depends upon its strength. It may be pointed out that maximum stabilization is achieved when the fluid is optically thick, the critically Rayleigh number is then $R_{MOC}(1+\chi)/D_1$ where R_{MOC} is the value of the Rayleigh number in the presence of magnetic field alone.

As would appear from (47) and (48), the conditions for the principle of exchange of stabilities to be true under approximation (a) are the same as in the absence of radiation.

The curves for $\log R_C$ versus $\log Q$ show the stabilizing effect of radiative transfer on the conducting fluid in the presence of a uniform magnetic field. This largely supports the conclusion obtained above. As remarked at the end of the last section, the presence of a very strong magnetic field lends a transparent character to the fluid. Also the effect of a very strong magnetic field for viscous and finitely conducting fluid (under approximation (a)) is to give rise to over-stability rather than convection first. These points seem to require further clarification.

From equation (58) it seems that the effect of radiative transfer on over-stability is twofold: (i) to reduce the effect of magnetic field on the Rayleigh number characterizing the marginal state for over-stability, and (ii) to stabilize the oscillations of increasing amplitude.

A similar problem which replaces magnetic field by rotation is under study. Preliminary investigations show that the results similar to (72) and (74) are not true in that case. The results of this investigation we hope to publish in the near future.

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Appendix

Derivation of equation (45) *from* (43) *and* (44)

Case (a). Multiply equation (43) by w_j (corresponding to n_j) and integrate with respect to ζ over the range $-\frac{1}{2} \leqslant \zeta \leqslant \frac{1}{2}$. After repeated integration by parts, we have

$$
\begin{aligned}
n_i^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw_i\, Dw_j + a^2 w_i\, w_j)\, d\zeta + n_i \bigg[(\nu+\kappa) \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2 w_i\, D^2 w_j + 2a^2 Dw_i\, Dw_j \\
+ a^4 w_i\, w_j)\, d\zeta + 3k^2 d^2 \chi\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw_i\, Dw_j + a^2 w_i\, w_j)\, d\zeta \bigg] + \bigg[\nu\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^3 w_i\, D^3 w_j \\
+ 3a^2 D^2 w_i\, D^2 w_j + 3a^4 Dw_i\, Dw_j + a^6 w_i\, w_j)\, d\zeta + 3k^2 d^2 \chi\nu\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2 w_i\, D^2 w_j \\
+ 2a^2 Dw_i\, Dw_j + a^4 w_i\, w_j)\, d\zeta + \gamma a^2 d^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\beta w_i\, w_j)\, d\zeta \bigg] \\
= \frac{\mu H_0 d}{4\pi\rho_0} \bigg[\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j (D^2 - a^2)^2 Dh_i\, d\zeta \\
- 3k^2 d^2 \chi\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j (D^2 - a^2)\, Dh_i\, d\zeta - n_i \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j (D^2 - a^2)\, Dh_i\, d\zeta \bigg]. \qquad \text{(I)}
\end{aligned}
$$

Multiplying both sides of equation (44) in terms of n_i and w_i by $(D^2-a^2)\,h_j$ and integrating by parts, we have

$$n_i \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dh_i\,Dh_j + a^2 h_i\,h_j)\,d\zeta + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta = H_0\,d \int_{-\frac{1}{2}}^{\frac{1}{2}} w_i(D^2-a^2)\,Dh_j\,d\zeta. \tag{II}$$

Interchanging i and j we obtain

$$n_j \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dh_i\,Dh_j + a^2 h_i\,h_j)\,d\zeta + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta = H_0\,d \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j(D^2-a^2)\,Dh_i\,d\zeta. \tag{III}$$

Multiplying (III) by n_i, we get

$$n_i\,n_j \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dh_i\,Dh_j + a^2 h_i\,h_j)\,d\zeta + \eta n_i \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta = H_0\,dn_i \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j(D^2-a^2)\,Dh_i\,d\zeta. \tag{IV}$$

Similarly, after multiplying (44) by $(D^2-a^2)^2\,Dh_j$, integrating by parts and then interchanging i and j, we get

$$n_j \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^3h_i\,D^3h_j + 3a^2 D^2h_i\,D^2h_j + 3a^4 Dh_i\,Dh_j + a^6 h_i\,h_j)\,d\zeta = -H_0\,d \int_{-\frac{1}{2}}^{\frac{1}{2}} w_j(D^2-a^2)^2\,Dh_i\,d\zeta. \tag{V}$$

Substituting from (III), (IV) and (V) in (I), we have

$$\begin{aligned}
&n_i^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw_i\,Dw_j + a^2 w_i\,w_j)\,d\zeta + n_i \bigg[(\nu+\kappa) \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2w_i\,D^2w_j + 2a^2 Dw_i\,Dw_j \\
&\quad + a^4 w_i\,w_j)\,d\zeta + 3k^2\,d^2\chi\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw_i\,Dw_j + a^2 w_i\,w_j)\,d\zeta\bigg] \\
&\quad + \nu\kappa \bigg[\int_{-\frac{1}{2}}^{\frac{1}{2}} (D^3w_i\,D^3w_j + 3a^2 D^2w_i\,D^2w_j + 3a^4 Dw_i\,Dw_j + a^6 w_i\,w_j)\,d\zeta \\
&\quad + 3k^2\,d^2\chi \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2w_i\,D^2w_j + 2a^2 Dw_i\,Dw_j + a^4 w_i\,w_j)\,d\zeta - Ra^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\beta/\bar{\beta})\,w_i\,w_j\,d\zeta\bigg] \\
&\quad + (\mu/4\pi\rho_0) \bigg[n_i\,n_j \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dh_i\,Dh_j + a^2 h_i\,h_j)\,d\zeta \\
&\quad + \eta n_i \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4\,h_i\,h_j)\,d\zeta \\
&\quad + n_j\bigg\{\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i Dh_j + a^4 h_i\,h_j)\,d\zeta + 3k^2 d^2\chi\kappa \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dh_i Dh_j + a^2 h_i\,h_j)\,d\zeta\bigg\} \\
&\quad + \eta\kappa \bigg\{\int_{-\frac{1}{2}}^{\frac{1}{2}} (D^3w_i\,D^3w_j + 3a^2 D^2w_i D^2w_j + 3a^4 Dw_i\,Dw_j)\,d\zeta \\
&\quad + 3k^2\,d^2\chi \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta\bigg\}\bigg] = 0.
\end{aligned} \tag{VI}$$

Interchanging i and j and then subtracting, we get equation (45).

Case (*b*). The corresponding equations for this case can easily be obtained from that of case (*a*) by letting $\chi \to 0$ and $\kappa \to \kappa_1$, where $\kappa_1 = \kappa(1+\chi)$. Thus in the final equation, corresponding to (45), interchanging i and j then subtracting, we get

$$(n_i - n_j)\left[(n_i+n_j)\int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw_i\,Dw_j + a^2 w_i\,w_j)\,d\zeta + (\nu+\kappa_1)\int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2w_i\,D^2w_j + 2a^2 Dw_i\,Dw_j + a^6 w_i\,w_j)\,d\zeta + (\mu/4\pi\rho_0)(\eta-\kappa_1) \times \int_{-\frac{1}{2}}^{\frac{1}{2}} (D^2h_i\,D^2h_j + 2a^2 Dh_i\,Dh_j + a^4 h_i\,h_j)\,d\zeta\right] = 0,$$

from which equation (62) can easily be derived.

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